AN INTEGRATED MODEL FOR PHYSICAL INTERPRETATION OF DEFORMATIONS

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Abstract

The integrated model to be presented allows for the physical interpretation of measured displacements including geodetic and mechanical relationships.
The general relationship between the variation principle of mechanics and the general case of the least squares adjustment will be shown. Thus the geodetic calculation methods can be applied favourably for the use of variation objectives. Applying the Lagrange function with multipliers leads to an extended model for the potential.

A complex deformation model based on the extended dynamical Hamilton's principle will be derived and recommended as an integrated solution procedure.

1. Introduction

The principles of virtual displacements or of the enforcement of stationarity for the total potential energy has been used for the formulations of the physical relations in the area of structure mechanics and solid body mechanics.
The solutions are achieved by applying Lagrangean functions. These functions represent the most general case. For specific applications special models and solutions are known which can all be derived from the general case.

The least square adjustment technology is the standard way to solve most problems for geodetic applications. It is of special interest to define and solve tasks in structural mechanics by making best use of the solution strategy, the specific properties and advantages of the least squares method. Therefore it seems necessary to describe the mechanical model in a general formulation and find an extension of the method of least squares to the most general case of correlated observables.
In the paper a specific way of solution is derived where two groups of Lagrangian factors are applied. Due to this general formulation it becomes apparent that the method of least squares and the application of the variation principle in mechanics lead to equivalent formulation. In both approaches the way to find an extremum via applying Lagrange factors.

Exploiting the analogy, the physical meaning of the parameters of the corresponding least squares problem become apparent. This results in a better understanding of the parameters, and the matrices of any least squares problem or mechanical variation problem. Most elaborate techniques of the least squares strategies, accuracy and reliability aspects can be applied to any variational application inf mechanics.

2. Geodetic least squares solution

If there are additional boundary conditions in order to find a minimum applying the method of least squares we use the Lagrangian function with Lagrangian factors. The Lagrangian function for a condition adjustment of observations with additional conditions between the unknowns reads, using observables l,misclosures w, residuals v and parameters x:

$$\underset{11}{\Phi} = \underset{1n}{v^T} \underset{nn}{Q_l^{-1}} \underset{n1}{v} - 2\underset{1r}{k_1^T}\left[\underset{rn}{B^T}\underset{n1}{v} + \underset{rm}{A}\underset{m1}{x} + \underset{r1}{w}\right] - 2\underset{1k}{k_2^T}\left[\underset{km}{G^T}\underset{m1}{x} + \underset{k1}{d}\right] \tag{2.0}$$

By setting the first derivative to zero and rearranging we get:

$$\begin{aligned} \frac{\partial \Phi}{\partial v} &= 2v^T Q_l^{-1} - 2k_1^T B^T = 0 \\ \frac{\partial \Phi}{\partial x} &= -2k_1^T A - 2k_2 G^T = 0 \end{aligned} \tag{2.1}$$

qonsequently:

$$\begin{aligned} & v^T Q_l^{-1} - k_1^T B^T = 0 \\ & A^T k_1 - G k_2 = 0 \end{aligned} \tag{2.2}$$

the residuals v read:

$$v = Q_l B k_1 \tag{2.3}$$

Substituting v in the condition equation and arranging the two condition equations together with the second equation of 2.2 we get:

$$\begin{aligned} B^T Q_l B k_1 \quad + Ax \qquad\qquad\qquad & + B^T l = 0 \\ A^T k_1 + G k_2 \qquad\qquad & = 0 \\ G^T x \qquad\qquad\qquad\qquad & + d = 0 \end{aligned} \tag{2.4}$$

In a squential elimination process the unknown parameters, Lagranian factors und the residuals can be determined.

2.1 Lagrangian factors k

From the first equation of 2.4 we receive:

$$k_1 = -N_1^{-1}Ax - N_1^{-1}B^T l \tag{2.5}$$

substituted to the second equation of 2.4 results in:

$$-A^T N_1^{-1}Ax - A^T N_1^{-1}B^T l + Gk_2 = 0 \tag{2.6}$$

or:

$$\begin{aligned} &Gk_2 - N_2 x - A^T N_1^{-1}B^T l = 0 \\ &G^T x + d = 0 \end{aligned} \tag{2.7}$$

The first equation multiplied by $G^T N_2^{-1}$ results in:

$$\begin{aligned} &G^T N_2^{-1}Gk_2 - G^T N_2^{-1}N_2 x - G^T N_2^{-1}A^T N_1^{-1}B^T l = 0 \\ &G^T x + d = 0 \end{aligned} \tag{2.8}$$

By adding the second term results in:

$$k_2 = N_3^{-1}G^T N_2^{-1}A^T N_1^{-1}B^T l - N_3^{-1}d \tag{2.9}$$

2.2 Unknown parameters

Substituting 2.9 in the first equation of 2.7 results in the formulation for x:

$$\begin{aligned} &x = N_2^{-1}\left(A^T N_1^{-1}B^T + GN_2^{-1}G^T N_2^{-1}A^T N_1^{-1}B^T\right)l - N_2^{-1}GN_3^{-1}d \\ &x = Dl + D_1 d \end{aligned} \tag{2.10}$$

Substituting x from (2.10) to (2.5) results in k_1:

$$\begin{aligned} k_1 = {} & [N_1^{-1}AN_2^{-1}(A^T N_1^{-1}B^T - GN_3^{-1}G^T N_2^{-1}A^T N_1^{-1}B^T) + N_1^{-1}B^T]l + \\ & + N_1^{-1}AN_2^{-1}GN_3^{-1}d \end{aligned} \tag{2.11}$$

2.3 Residuals:

In the same way the residuals of 2.3 result by substituting k_1 from 2.5:

$$v = Q_l B N_1^{-1}(AN_2^{-1}AN_1^{-1}B^T - AN_2^{-1}GN_3^{-1}G^T N_2^{-1}AN_1^{-1}B^T + B^T)l + \\ + N_1^{-1}AN_2^{-1}GN_3^{-1}D \tag{2.12}$$

3. Mehanical solution

The variational method finds one of most fruitful fields of application in the small displacement theory of elasticity. When the existence of a strain energy function is assured and the external forces are assumed to be kept unchanged during displacement variation, the principle of virtual work leads to the establishment of the principle of minimum potential energy Π. If the displacements are calculated, the onother variables like strain and stress values can be dyrectly obtained from the equations (3.5) and (3.10). Approximate values for the displacements can be obtained by using the model given in (Milev, Gruendig 1999).

According to (Bathe 1986) a finite element solution can be formulated in the following :

3.1 The potential energie

The form

$$\Pi^{(D)} = \Pi_i - \Pi_a \tag{3.1}$$

describes the complete potential energie with internal potential stated matematically as follows:

$$\Pi_i = \frac{1}{2}\int_B \left[\varepsilon - \bar{\varepsilon}_T\right]^T C \left[\varepsilon - \bar{\varepsilon}_T\right] dB = \int_B \delta\varepsilon^T \sigma \; dB \tag{3.2}$$

and the potential of the external forces:

$$\Pi_a = \int_B u^T \bar{p} \cdot dB + \int_{R_\sigma} u_r^T \bar{q}_R \cdot dR \; . \tag{3.3}$$

The constraints of the stationarity of Π with respect to the displacements leads to the formulation of equilibrium:

$$\int_B \partial\varepsilon^T C\,\varepsilon dB = \int_B \partial u^T \bar{p}\cdot dB + \int_{R_\sigma} \partial u_r^T \bar{q}_R \cdot dR \tag{3.4}$$

The Strain-displacement conditions of compatibility (3.5) and the displacement boundary conditions (3.6) will be fulfilled exactly

$$\varepsilon = BU \tag{3.5}$$

$$U_a - U^p = 0 \tag{3.6}$$

where:

U^p - prescribed terms of displacement,

U_a - displacements components of U.

The constraints for stationarity is the compatibility (3.5) and after fulfilling (3.6) results the equilibrium.

The presented displacement related finit element theorie is very common in the praxis. However, other methods have also been applied sucessfully. This will be explained in the following.

3.2 Stationarity of the complementary energy

One of those methods is based on the stationarity of the complementary energy.

$$\Pi^* = \frac{1}{2}\int_B \tau^T C^{-1}\,\tau dB - \int_{R_\sigma} \partial u_r^T \bar{\sigma}_R \cdot dR \tag{3.7}$$

The transition from the potential energy to the supplemental energy can be achieved by a Legendre-transformation. The supplemental energy is obtained by maximizing the energy. The area integral will be formed for a surface R in addition to the constraint for some boundary conditions. The stress components and the corresponding loads on the surface are modelled by unknown parameters. Simultaneously the stress functions have to fulfil the continuity between the elements, the differential equilibrium conditions and the boundary conditions for stress. The condition for stationarity of Π^* with respect to the stress parameters results in:

$$\int_B \tau^T C^{-1}\,\tau dB = \int_{R_\sigma} \partial u_r^T \bar{\sigma}_R \cdot dR \tag{3.8}$$

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and the equations for equilibrium of stress

$$\bar{\sigma} = Rot\ Rot\ \bar{\phi} \tag{3.9}$$

$$\overline{\sigma_r} - k = 0 \tag{3.10}$$

The requirements for the stationarity is the equilibrium condition (3.9), and from (3.10) results the compatibility condition.

It is obious that some conditions are fulfilled exactly, however this is not completely true for the strain, stress and compatibility conditions and the geometrical boundary conditions. They will only be fulfilled if (3.8) will give be a satisfactory solution for any variation of stresses. The fomulation of the stationarity of the supplement enery has not widely been applied in practice as the setting up of the stress function $\bar{\phi}$ has proven to be difficult for many practical applications.

3.3 Gemischte Variationsformulierungen

Mixed variations are expressions which result from an extension of the principle of stationarity of the total potential energy or of the suplemental enery. They allow for a relaxation of the conditions which have to be fulfilled by the solution variables. They also allow for additonal solution variables which might represent deflections, strain or stress. Starting from (3.1) until (3.6) the conditions for displacements following (3.5) and (3.6) can be weakened by applying additional Lagrangian multipliers in Π. The functional formulation reads:

$$\prod{}_1 = \Pi + \int_B \lambda_\varepsilon^T (\varepsilon - BU) dB + \int_{R_u} \lambda_u^T (U_a - U^p) dR \tag{3.11}$$

where λ_ε and λ_u are Lagrangesche factors, U^p the prescribed boundary deflections at the border, and R the complete surface. λ_ε and λ_u an be seen as stress and forces.

The formulation of the variations shown above may be seen as a generalisation of the principle of virtual displacements. The boundary condition for the displacements and the compatibility conditions for strain have been relaxed. All unknown displacements, strain parameters, stresses and boundary forces vary. A formulation of the type (3.11) is a valuable and most general description of the statical and kinematic condition of the body under consideration. Enforcing stationarity with respect to every unknown variable ends in the following equation for the material law,

$$\tau = C\varepsilon \tag{3.12}$$

the compatibility conditions (Strain-displacement relations):

$$\varepsilon = BU \tag{3.13}$$

and the equations of equilibrium:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x^B = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y^B = 0 \qquad (3.14)$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z^B = 0$$

for the volums B_V. On the boundary of the body the prescribed displacements R_u and force conditions R_f will be enforced. The internal reactions with respect to R_u are identical to the Lagrangian factors.

4. **Analogy between the Lagrangian formulations of geodetic adjustment and mechanics**

In 2.2 and 2.3. the general case of least squares adjustment and the principles of variations have been formulated. In both cases the same starting situation is given, the search for an extreme of a function applying the Lagrangian formulation. As stated in 2.1 the analogy between the functional representations is worth considering, and especially with respect to the variables and matrices used. Based on that analogy it will be easier to solve the formulations of variations in mechanics, namely based on the method of least squares.

By comparing both formulations a number of characteristics become obvious.

$$\Pi_1 = \Pi + \int_B \lambda_\varepsilon^T (\varepsilon - BU) dB + \int_{R_u} \lambda_u^T (U_a - U^p) dR$$

$$\underset{11}{\Phi} = \underset{1n}{v^T} \underset{nn}{Q_l^{-1}} \underset{n1}{v} - 2 \underset{1r}{k_1^T} \left[\underset{rn}{B^T} \underset{n1}{v} + \underset{rm}{A} \underset{m1}{x} + \underset{r1}{w} \right] - 2 \underset{1k}{k_2^T} \left[\underset{km}{G^T} \underset{m1}{x} + \underset{k1}{d} \right]$$

Π corresponds to sum of the squared residuals $v^T Q^{-1} v$. The expression $\varepsilon - BU$, the geometrical compatibility condition corresponds to the condition equations of the least squares formulation $Bv + Ax + w$. The displacement conditions $U_a - U^p$ correspond to the additional conditions between the unknown parameters $Gx + d$. The Lagangian multipliers λ_1, λ_2 are k_1 , k_2 in both sets of equations. They reduce the constraints to forces and corresponding stresses.

The displacements and therefore the strain ε will vary, which can be interpreted as residuals v. For the unknown deflections U_a good approximate values can be obtained following the procedure described in (Milev and Gruendig 1999). The can be substituted within the parameters x of the unknowns, where any set of given displacements U^p corresponds to the parameters d .
For the physical interpretation of deflections in a deformation analysis, the parameters d may result from a comparison of epochs of geodetic measurements (Gruendig et al.

1985). They are the statistically significant parameters. The interpretation of the mechanical model, where stress and strain reaction are to be determined can easily be done (Milev 2001).

5. **Dynamical model**

Hamilton‘s prinziple for a elastic body in his dynamical formulation states:
Among the set of all admissible configurations of the system, the actual motion makes the quantyty

$$\delta \int_{t_1}^{t_2} L dt = 0 , \tag{5.1}$$

stationary, provided the configoration of the system is prescribed at the limits $t = t_1$ and $t = t_2$, where $L = T - U$ is a extended expresion of the Lagrangian function given in (3.11). T meens the kinetic energy of the system.

6. Conclusions

In this paper a comparison between the general relations of variation in mechanics and the general solution of the least squares problem of the type conditional adjustment, subject to conditions between the unknown parameters, has been explained.

For the principle of variations, subject to conditions, the solution will be achieved applying Lagrangian factors. Combing the characteristics of the least squares approach in Geodesy with the physical meaning of the parameters in mechanics, a better understanding and a physical interpretation of surveying results will be obtained. For the mechanical application, stochastic properties of the parameters can be derived and used for a verification of the results.
The extended Hamilton‘sche functional seems to most aprobriate for the general case of deformation analysis, as it also includes time.

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